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Aufgabe 1

a) $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$; $(i\not\partial - m)\psi = 0$ $\quad | \cdot$ links $(i\not\partial - m)$

$(i\not\partial - m)(i\not\partial - m)\psi = 0$

$-\not\partial\not\partial\psi \underbrace{-2im\not\partial\psi}_{D\text{-Gl.}} + m^2\psi = 0. \quad | \Rightarrow -\not\partial\not\partial\psi - 2mi^2\psi + m^2\psi = 0 \quad (*)$

$\not\partial\not\partial\psi = (\gamma^\mu\gamma^\nu)\partial_\mu\partial_\nu\psi = (2g^{\mu\nu} - \gamma^\nu\gamma^\mu)\partial_\mu\partial_\nu\psi =$

$= 2\partial^\mu\partial_\mu\psi - \not\partial\not\partial\psi \Rightarrow \not\partial\not\partial\psi = \partial^\mu\partial_\mu\psi = \Box\psi \quad (**)$

$\xRightarrow{(*)(**)} (\Box + m^2)\psi = 0 \Rightarrow (\Box + m^2)\psi_i = 0, \; i = 1,4$

b) $Tr(\not p\not p \underbrace{\gamma^\mu\gamma_\mu}_{4}) = 4Tr(\not p\not p) = 4p_\alpha p_\beta Tr(\gamma^\alpha\gamma^\beta) =$

$= 4p_\alpha p_\beta \cdot 4g^{\alpha\beta} = 16(p\cdot p)$

c) $a^\mu \to a'^\mu = \Lambda^\mu{}_\alpha a^\alpha$

$b^\mu \to b'^\mu = \Lambda^\mu{}_\beta b^\beta$ $\quad | \Rightarrow T^{\mu\nu} = a^\mu b^\nu \to T'^{\mu\nu} = a'^\mu b'^\nu$

$T'^{\mu\nu} = \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta \underbrace{a^\alpha b^\beta}$ $=$

$= \underbrace{\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta} T^{\alpha\beta}.$

$\Rightarrow T^{\mu\nu}$ = Tensor 2. Stufe

d) $1/2 \otimes 1/2 = 1 \oplus 0$. Sei $|m_{s_1}, m_{s_2}\rangle = |m_1\rangle \otimes |m_{s_2}\rangle$

$S=0 \Rightarrow \chi_{S=0}(\vec s_1, \vec s_2) = \frac{1}{\sqrt 2}(|+,-\rangle - |-,+\rangle)$ = anti-symmetrisch.

$\Rightarrow \boxed{\psi(\vec r_1, \vec s_1; \vec r_2, \vec s_2) = \psi_R(\vec r_1, \vec r_2)\,\chi_S(\vec s_1, \vec s_2)}$ $\Rightarrow$

anti-symmetrisch $\qquad$ anti-symmetrisch

$\psi_R(\vec r_1, \vec r_2)$ = symmetrisch in $(\vec r_1, \vec r_2)$ $\Rightarrow$

$\psi_R(\vec r_1, \vec r_2) = \frac{1}{\sqrt 2}(\psi_1(\vec r_1)\psi_2(\vec r_2) + \psi_1(\vec r_2)\psi_2(\vec r_1))$.

e) $Tr(\gamma^\alpha\gamma^\beta\gamma^\sigma) = 0$.

i) $\alpha = \beta = \sigma \Rightarrow Tr(\underbrace{\gamma^\alpha\gamma^\alpha}_{\mathbb{1}}\gamma^\alpha) = Tr(\gamma^\alpha) = 0$

ii) $\underbrace{\alpha = \beta \neq \sigma}_{\{\gamma^\alpha, \gamma^\sigma\} = 0}$ oder $\alpha = \sigma \neq \beta \Rightarrow Tr(\gamma^\alpha\gamma^\beta\gamma^\sigma) = \begin{cases} Tr(\gamma^\sigma) = 0 \\ -Tr(\gamma^\beta) = 0 \end{cases}$.

iii) $\alpha \neq \beta \neq \sigma \Rightarrow Tr(\gamma^\alpha\gamma^\beta\gamma^\sigma) = Tr(\gamma^\alpha\gamma^\beta\gamma^\sigma\overbrace{(\gamma^\mu)^2}^{\mathbb{1}})$ mit $\alpha \neq \beta \neq \sigma \neq \mu \in$

$= Tr(\gamma^\mu\gamma^\alpha\gamma^\beta\gamma^\sigma\gamma^\mu) = (-1)^3 Tr(\underbrace{(\gamma^\mu)^2}_{\mathbb{1}}\gamma^\alpha\gamma^\beta\gamma^\sigma) = \underline{-Tr(\gamma^\alpha\gamma^\beta\gamma^\sigma)}$

$\Rightarrow Tr(\gamma^\alpha\gamma^\beta\gamma^\sigma) = 0$.

f) $\bar\gamma^\mu = \gamma^0(\gamma^\mu)^\dagger\gamma^0 = \begin{cases} \mu = 0 \Rightarrow \bar\gamma^0 = \gamma^0\gamma^0\gamma^0 = \gamma^0 \\ \mu = i \Rightarrow \bar\gamma^i = \gamma^0(-\gamma^i)\gamma^0 = +(\gamma^0)^2\gamma^i = \gamma^i \end{cases}$

$(\gamma^0)^\dagger = (\beta^\dagger) = \beta = \gamma^0$

$(\gamma^i)^\dagger = (\beta\alpha^i)^\dagger = (\alpha^i)^\dagger(\beta^\dagger) = \alpha^i\beta = -\beta\alpha^i = -\gamma^i$

g) $\psi'(x') = S(\Lambda)\psi(x)$ $\quad$ mit $S\gamma^\mu S^{-1}\Lambda^\nu{}_\mu = \gamma^\nu$

$\bar\psi'(x') = \psi^{\dagger\prime}(x')\gamma^0 = (S(\Lambda)\psi(x))^\dagger\gamma^0 = \psi^\dagger(x)S^\dagger(\Lambda)\gamma^0$.

$\Rightarrow \underline{\bar\psi'\gamma^\mu\psi'} = \underbrace{\psi^\dagger\gamma^0}_{\bar\psi}\gamma^0 S^\dagger\gamma^0\gamma^\mu S\psi = \bar\psi\,\underbrace{(\gamma^0 S^\dagger\gamma^0)}_{S^{-1}}\gamma^\mu S\psi =$

{aus: $S\gamma^\mu S^{-1}\Lambda^\nu{}_\mu = \gamma^\nu \Rightarrow S^{-1}\gamma^\nu S = \Lambda^\nu{}_\mu\gamma^\mu$

$= \bar\psi\,\Lambda^\mu{}_\alpha\gamma^\alpha\psi = \underline{\Lambda^\mu{}_\alpha(\bar\psi\gamma^\alpha\psi)}$

$\Rightarrow \bar\psi\gamma^\mu\psi$ ein Vektor.

h) $(\exists)$ nur $\underline{3}$ anti-kommutierende dim=2 Matrizen (die Pauli-Matrizen $\sigma_1, \sigma_2, \sigma_3$). Für $\{\gamma^\alpha, \gamma^\beta\} = 2g^{\alpha\beta}$ braucht man $\underline{4}$ anti-kommutierende Matrizen.

i) Dipolnäherung: $\Delta l = \pm 1$; $\Delta m_l = 0, \pm 1$; $(\Delta m_s = 0)$

$2P \to 1S$ $\qquad$ $2P: |n=2, l=1, m_l = -1, 0, 1\rangle \Rightarrow$

$1S \quad |n=1, l=0, m_l = 0\rangle$

$|2,1,m_l=+1\rangle \to |1,0,m_l=0\rangle$

$|2,1,m_l=0\rangle \to |1,0,m_l=0\rangle$

$|2,1,m_l=-1\rangle \to |1,0,m_l=0\rangle$.

alle Übergänge sind erlaubt.

) $i\hbar\,\partial_t \psi_I = V_I(t)\psi_I$ mit $V_I(t) = e^{iH_0 t/\hbar}\, V(t)\, e^{-iH_0 t/\hbar}$

$$\psi_I(t) = \psi_I(t_0) + \frac{1}{i\hbar}\int_{t_0}^{t} dt'\, V_I(t')\,\psi_I(t')$$

c) $1 \otimes 1 \Rightarrow \vec F = \vec S_1 + \vec S_2$; $F = 2;\ 1;\ 0$
$M_F = (2F+1) = 5\ ;\ 3\ ;\ 1$ = Entartungsgrad.

Aufgabe 2 $\quad H = H_0 + A\,\vec S_1\cdot\vec S_2 + (-\vec\mu_1\cdot\vec B - \vec\mu_2\cdot\vec B)$

mit $\vec\mu_i = \gamma_i\,\vec S_i$

Matrixelement des Störoperators $H_{HF} + H_Z$ in der Basis $\{|F, m_F\rangle\}$
wobei $\vec F = \vec S_1 + \vec S_2$ = Gesamtspin.

$\frac12 \otimes \frac12 = 1 \oplus 0$. ; Seien $|m_S\rangle_1$ und $|m_S\rangle_2$ die Zust. des e^-_S bzw. des μ^+_S.

$F=1 \Rightarrow |1,+1\rangle = |+\rangle_1|+\rangle_2$

$|1,0\rangle = \frac{1}{\sqrt2}\left(|+\rangle_1|-\rangle_2 + |-\rangle_1|+\rangle_2\right)$

$|1,-1\rangle = |-\rangle_1|-\rangle_2$

$F=0 \quad |0,0\rangle = \frac{1}{\sqrt2}\left(|+\rangle_1|-\rangle_2 - |-\rangle_1|+\rangle_2\right)$.

$\Rightarrow \vec S_1\cdot\vec S_2$ diagonal in der Basis $\{|F, m_F\rangle\}$, da

$$\vec S_1\cdot\vec S_2 = \frac{\vec F^2 - \vec S_1^2 - \vec S_2^2}{2} = \frac{\hbar^2}{2}\left[F(F+1) - S_1(S_1+1) - S_2(S_2+1)\right] =$$

$$= \begin{cases} \frac{\hbar^2}{4} & ; F = 1 \\ -\frac34\hbar^2 & ; F = 0 \end{cases}$$

$$H_Z = -(\underbrace{\gamma_1 B_z}_{-\omega_1} S_{1,z} + \underbrace{\gamma_2 B_z}_{-\omega_2} S_{2,z}) \equiv +\ \omega_1 S_{1,z} + \omega_2 S_{2,z}$$

$\Rightarrow (H_{HF} + H_Z)_{\{|F,m_F\rangle\}} =$; $\{|F,m_F\rangle\} = \{|1,1\rangle, |1,0\rangle, |1,-1\rangle, |0,0\rangle\}$

$$\begin{pmatrix} \frac{A\hbar^2}{4} + \frac{\hbar}{2}(\omega_1+\omega_2) & 0 & 0 & 0 \\ 0 & \boxed{\frac{A\hbar^2}{4}} & 0 & \boxed{\frac{\hbar}{2}(\omega_1-\omega_2)} \\ 0 & 0 & \frac{A\hbar^2}{4} - \frac{\hbar}{2}(\omega_1+\omega_2) & 0 \\ 0 & \frac{\hbar}{2}(\omega_1-\omega_2) & 0 & -\frac34 A\hbar^2 \end{pmatrix}$$

$\Delta E_1^{(1)} = \frac{A\hbar^2}{4} + \frac{\hbar}{2}(\omega_1+\omega_2)$; $\Delta E_3^{(1)} = \frac{A\hbar^2}{4} - \frac{\hbar}{2}(\omega_1+\omega_2)$

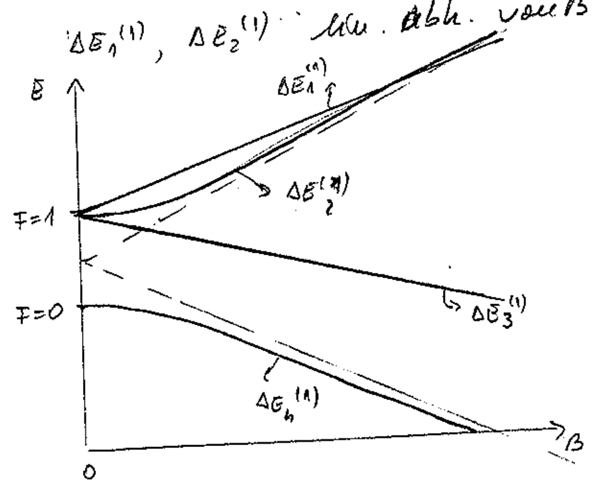
Im Unterraum $\{|1,0\rangle; |0,0\rangle\}$ diagonalisiere 2×2 Matrix

$$\Rightarrow \Delta E_{2,4}^{(1)} = -\frac{A\hbar^2}{4} \pm \sqrt{\left(\frac{A\hbar^2}{2}\right)^2 + \left(\frac{\hbar(\omega_1-\omega_2)}{2}\right)^2} =$$

$$= -\frac{A\hbar^2}{4} \pm \sqrt{\left(\frac{A\hbar^2}{2}\right)^2 + \frac{\hbar^2(\gamma_1-\gamma_2)^2 B^2}{4}}$$

b) $B \to 0 \Rightarrow \Delta E_1^{(1)} = \Delta E_3^{(1)} = \frac{A\hbar^2}{4}$; $\Delta E_2^{(1)} = +\frac{A\hbar^2}{4}$; $\Delta E_4^{(1)} = -\frac34 A\hbar^2$

$\Delta E_1^{(1)}$, $\Delta E_2^{(1)}$ lin. abh. von B.



Aufgabe 3 $\quad \mathcal{M} = [\bar u(p_f)\gamma^\mu u(p_i)]\frac{e^2}{q^2}[\bar u(P_f)\gamma_\mu u(P_i)] =$

$$\overline{|\mathcal{M}|^2} = \frac14 \sum_{s_i, s_f, S_i, S_f} \left|\bar u(p_f)\gamma^\mu u(p_i)\,\frac{e^2}{q^2}\,\bar u(P_f)\gamma_\mu u(P_i)\right|^2 =$$

$$= \frac14 \sum_{s_i,s_f,S_i,S_f} \bar u_\alpha(p_f,s_f)\gamma^\mu_{\alpha\beta}\,\underbrace{u_\beta(p_i,s_i)\,\bar u_\sigma(p_i,s_i)}\,\gamma^\nu_{\sigma\delta}\,u_\delta(p_f,s_f)\ \frac{e^2}{q^2}$$
$$\bar u(P_f,S_f)\gamma_\mu\,\underbrace{u(P_i,S_i)\,\bar u(P_i,S_i)}\,\gamma_\nu\,u(P_f,S_f)$$

$$= \frac14\, Tr\left[\frac{\not p_f + m}{2m}\gamma^\mu \frac{\not p_i + m}{2m}\gamma^\nu\right]\frac{e^2}{q^2}\, Tr\left[\frac{\not P_f + M}{2M}\gamma_\mu\frac{\not P_i + M}{2M}\gamma_\nu\right]$$

$$= \frac{1}{64}\,\frac{e^2}{m^2M^2q^2}\, Tr\left[(\not p_f + m)\gamma^\mu(\not p_i + m)\gamma^\nu\right] Tr\left[(\not P_f + M)\gamma_\mu(\not P_i + M)\gamma_\nu\right].$$

$$\mathrm{Tr}\left[(\not p_f+m)\gamma^\mu(\not p_i+m)\gamma^\nu\right]=p_{f\alpha}p_{i\beta}\left[4\left(g^{\alpha\mu}g^{\beta\nu}+g^{\alpha\nu}g^{\mu\beta}-g^{\alpha\beta}g^{\mu\nu}\right)+4m^2g^{\mu\nu}\right]=4\left[p_f^\mu p_i^\nu+p_f^\nu p_i^\mu-(p_i\cdot p_f)g^{\mu\nu}+m^2g^{\mu\nu}\right].$$

$$\mathrm{Tr}\left[(\not P_f+M)\gamma_\mu(\not P_i+M)\gamma_\nu\right]=4\left[P_{f\mu}P_{i\nu}+P_{f\nu}P_{i\mu}-(P_i\cdot P_f)g_{\mu\nu}+M^2g_{\mu\nu}\right]$$

$$\Rightarrow \mathrm{Tr}[\;]\cdot\mathrm{Tr}[\;]=16\cdot\left[2(P_f\cdot p_f)(P_i\cdot p_i)+2(P_f\cdot p_i)(P_i\cdot p_f)-\cancel{2(p_i\cdot p_f)}(P_i\cdot P_f)+\cancel{2m^2}(P_i\cdot P_f)-2\cancel{(p_i\cdot p_f)}(P_i\cdot P_f)+4(P_i\cdot P_f)(p_i\cdot p_f)-\underset{2}{\cancel{4}}(P_i\cdot P_f)\,m^2+\cancel{2M^2}(p_i\cdot p_f)-\underset{2}{\cancel{4}}M^2(p_i\cdot p_f)+M^2m^2\cdot 4\right]=$$

$$=16\cdot 2\left[(P_f\cdot p_f)(P_i\cdot p_i)+(P_f\cdot p_i)(P_i\cdot p_f)-m^2(P_i\cdot P_f)-M^2(p_i\cdot p_f)+2M^2m^2\right].$$

$$\Rightarrow \overline{|\mathcal{M}|^2}=\frac{1}{2}\frac{e^4}{m^2M^2q^4}\left[(P_f\cdot p_f)(P_i\cdot p_i)+(P_f\cdot p_i)(P_i\cdot p_f)-m^2(P_f\cdot P_i)-M^2(p_i\cdot p_f)+2M^2m^2\right].\quad (**)$$

b) $q^2=(p_f-p_i)^2=(P_i-P_f)^2$

Laborsystem: $p_i=\begin{pmatrix}E\\\vec p\end{pmatrix}$; $p_f=\begin{pmatrix}E'\\\vec p\,'\end{pmatrix}$; $\vec p\cdot\vec p\,'=|\vec p||\vec p\,'|\cos\theta$

$p_i^2=E^2-\vec p^{\,2}=m^2$; $\frac{m}{E}\ll 1\Rightarrow$ $\vec p^{\,2}=E^2$ $\Big|\Rightarrow$
$p_f^2=(E')^2-|\vec p\,'|^2=m^2$; $(\vec p\,')^2=(E')^2$

$q^2=(p_f-p_i)^2=(E-E')^2-(\vec p-\vec p\,')^2=(E-E')^2-(E^2+E'^2-2EE'\cos\theta)$
$=-2EE'(1-\cos\theta)=\underline{-4EE'\sin^2(\theta/2)}$.

$P_i=\begin{pmatrix}M\\0\end{pmatrix}$; $P_f=p_i+P_i-p_f\quad\Rightarrow P_f^2=(p_i-p_f+P_i)^2\Rightarrow$

$\Rightarrow M(E'-E)+EE'-|\vec p||\vec p\,'|\cos\theta=m^2$; $m^2\ll E^2\Rightarrow$

$\boxed{M(E-E')=EE'(1-\cos\theta)}$ ⊕

$\underline{p_i\cdot P_i=E\cdot M}$; $\underline{p_f\cdot P_f=E'\cdot M}$; $\underline{(p_i\cdot p_f)=EE'(1-\cos\theta)}$

$(p_i-P_f)=(p_f-P_i)\Rightarrow p_i\cdot P_f=p_f\cdot P_i$; $p_f\cdot P_f=p_i\cdot P_i$

$\Rightarrow$ einsetzen in $(**)$

$$\overline{|\mathcal{M}|^2}=\frac{e^4}{2m^2M^2(-4EE'\sin^2\theta/2)^2}\cdot\left[(E\cdot M)^2+(E'M)^2-m^2(M^2-m^2+EE'(1-\cos\theta))-M^2EE'(1-\cos\theta)+\cancel{2}m^2M^2\right]\quad\{\mathcal{O}(m^2)\to 0.\}\Rightarrow$$

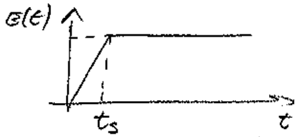
$$=\frac{e^4}{2m^2E^2E'^2\sin^4(\theta/2)}\Big[\underbrace{E^2+E'^2}_{\text{aus ⊕}}-EE'\underbrace{(1-\cos\theta)}_{2\sin^2\theta/2}\Big]$$

$$=\frac{e^2}{2m^2EE'\sin^4(\theta/2)}\left[\cancel{2}-\frac{q^2}{M^2}\sin^2(\theta/2)-\cancel{1}+\cos\theta\right]=$$

$$=\frac{e^2}{m^2EE'\sin^4(\theta/2)}\left[\cos^2\frac{\theta}{2}-\frac{q^2}{2M^2}\sin^2(\theta/2)\right]$$

mit $q^2=-4EE'\sin^2\theta/2$;

Aufgabe 4   Sudden Approximation



• Ursprünglicher stationärer Zust. von H_0:

$|n^{(0)} t\rangle = e^{-iE_n^{(0)}t/\hbar}\,|n^{(0)}\rangle$

$t_s \ll t_{ch}$ = char. Zeit der Umordnung des Syst.

• Stationäre Zust. von H_0+V:

$|\bar n, t\rangle = e^{-i\bar E_n t/\hbar}\,|\bar n\rangle$

• $t \le 0$ $\quad |\Psi, t\rangle = \phi_0(x)\, e^{-iE_0^{(0)}t/\hbar}$ $\quad ; E_0^{(0)} = \frac{\hbar\omega}{2}$;

$t=0_+$ $|\bar\Psi, 0\rangle = |\Psi, 0\rangle$ unmittelbar nach dem Einschalten des Potentials befindet sich das Syst. in diesem Zust.

$t>0$: $V(x) = \frac{m\omega^2 x^2}{2} - qE\cdot x = \frac{m\omega^2}{2}\left(x^2 - \frac{2E\varepsilon}{m\omega^2}x + \frac{E^2\varepsilon^2}{m^2\omega^4}\right) - \frac{E^2\varepsilon^2}{2m\omega^2}$

$= \frac{m\omega^2}{2}(x-x_0)^2 + V_0$ min $\quad x_0 = \frac{E\varepsilon}{m\omega^2}$; $V_0 = -\frac{E^2\varepsilon^2}{2m\omega^2}$

$H(x) = \frac{p^2}{2m} + \frac{m\omega^2}{2}(x-x_0)^2 + V_0 =$ $\qquad y = x - x_0.$

$= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{m\omega^2}{2}(x-x_0)^2 + V_0$

$H(y) = -\frac{\hbar^2}{2m}\frac{d^2}{dy^2} + \frac{m\omega^2}{2}y^2 + V_0.$

Gesucht $\quad H(y)\,\bar\Psi(y) = \bar E\,\bar\Psi(y)$

$\Rightarrow \bar\Psi(y) = \phi_n(y)$ $\quad ; \bar E_n = V_0 + E_n = V_0 + \hbar\omega(n+\frac{1}{2})$

$\Rightarrow \boxed{\bar\Psi_n(x) = \phi_n(x-x_0) = \mathcal{C}_n H_n\left(\frac{x-x_0}{\rho_0}\right) e^{-\frac{(x-x_0)^2}{2\rho_0^2}}}$ $\quad \mathcal{C}_n = (2^n n!\sqrt{\pi}\rho_0)^{1/2}$

$\begin{cases} \bar\Psi(x,t) = \sum_n c_n(t)\,\bar\Psi_n(x) & \text{mit} \\ \bar\Psi(x,t=0) = \phi_0(x) & (t=0) \end{cases} \Rightarrow$

$c_n(0) = \langle\phi_0(x)|\,\phi_n(x-x_0)\rangle = \int_{-\infty}^{+\infty} dx\, \phi_n(x-x_0)\,\phi_0^*(x) =$

$= \rho_0^{-1}(\pi 2^n n!)^{-1/2}\int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}\left(\frac{x}{\rho_0}\right)^2 - \frac{1}{2}\left(\frac{x-x_0}{\rho_0}\right)^2\right) H_n\left(\frac{x-x_0}{\rho_0}\right) dx$

$\xi = (x-x_0)/\rho_0 \quad \Rightarrow$

(Hinweis)

$c_n(0) = (\pi 2^n n!)^{-1/2}\int_{-\infty}^{+\infty} d\xi\, H_n(\xi)\exp\left[-\left(\xi^2 + \frac{1}{2}(x_0/\rho_0)^2 + \xi(-x_0/\rho_0)\right)\right]$

$= \frac{(-1)^n \left(\frac{x_0}{\rho_0}\right)^n \exp\left(-\frac{1}{4}\frac{x_0^2}{\rho_0^2}\right)}{\sqrt{2^n n!}}$ $\quad ; \ c_n(t) = c_n(0)\, e^{-i\bar E_n t/\hbar}$

$P_{\Psi_0\to\bar n} = |\langle\bar n|\Psi,0\rangle|^2 = |c_n|^2 = |c_n(0)|^2 = \frac{\left(\frac{x_0}{\rho_0}\right)^{2n}\exp\left(\frac{-x_0^2}{2\rho_0^2}\right)}{2^n n!}$

↳ Poisson Distribution