A 1. /Variante A. / B

a) $H_S = -\varepsilon E \cdot z$ E.Ven von H_0 $\{|n,\ell,m\rangle\}$

$$\Delta G^{(1)} = \langle 1,0,0|-\varepsilon E z|1,0,0\rangle = \int r^2 dr\, R_{10}^2(r) \cdot r \underbrace{\int d\Omega\, (Y_{00})^2\, Y_{10}}_{0} =$$

G.Z symm.   anti-symm. $(\vec{r} \to -\vec{r})$ $z \sim Y_{10}(\theta,\varphi)\cdot r$.

b) i) $\vec{a}\cdot\vec{b}$ = Skalar ⇒ inv unter räuml. Drehungen.

ii) $a_0 b_0$ = Skalar " ——— "; $\begin{cases} a_0' b_0' = \Lambda_0{}^\mu a_\mu \Lambda_0{}^\nu b_\nu = a_0 b_0 \\ \text{räuml. Drehungen } \Lambda_0{}^\mu = \begin{cases} 1, & \mu = 0 \\ 0, & \text{sonst} \end{cases} \end{cases}$

Variante B.

i) $a_0 \vec{b} \longrightarrow a_0' (b^j)' = \Lambda_0{}^\mu a_\mu \Lambda^j{}_\nu b^\nu = a_0 \Lambda^j{}_\nu b^\nu = a_0 \underline{\Lambda^j{}_i} b^i$

↳ Vektor; nicht inv.

ii) $a_\mu b_\nu c_\rho \epsilon^{\mu\nu\rho 0} = a_\mu b_\nu c_\rho \epsilon^{\mu\nu\rho} = a_\mu (\epsilon^{\mu\nu\rho} b_\nu c_\rho)$

$\epsilon^{\mu\nu\rho 0} = \begin{cases} \neq 0 & \mu \neq \nu \neq \rho \neq 0 \Rightarrow \mu,\nu,\rho \in 1,2,3 \\ 0 & \text{sonst} \end{cases}$

$= a_\mu (\vec{b}\times\vec{c})^\mu = \vec{a}\cdot(\vec{b}\times\vec{c})$ = Skalar, inv. u. räuml. Drehungen.

c) $\gamma^\mu \not{a} \not{b} \gamma_\mu = a_\alpha b_\beta \gamma^\mu \gamma^\alpha \gamma^\beta \gamma_\mu = a_\alpha b_\beta (2g^{\mu\alpha} - \gamma^\alpha\gamma^\mu)\gamma^\beta\gamma_\mu$

$= 2a^\mu b_\beta \gamma^\beta \gamma_\mu - a_\alpha b_\beta \gamma^\alpha (2g^{\mu\beta} - \gamma^\beta\gamma^\mu)\gamma_\mu =$

$= 2\not{b}\not{a} - 2\not{a}\not{b} + a_\alpha b_\beta \gamma^\alpha\gamma^\beta \underbrace{\gamma^\mu\gamma_\mu}_{4} = 2(\not{a}\not{b} + \not{b}\not{a}) =$

$= 2a_\alpha b_\beta (\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha) = 2a_\alpha b_\beta \cdot 2g^{\alpha\beta} = 4\, a\cdot b$.

Variante B

$\gamma^\mu \gamma^\alpha \gamma^\beta \gamma_\mu = 2\gamma^\beta\gamma^\alpha - 2\gamma^\alpha\gamma^\beta + 4\gamma^\alpha\gamma^\beta = 2\{\gamma^\alpha, \gamma^\beta\} = 4g^{\alpha\beta}$

d) $[\gamma^\mu, \sigma^{\nu\rho}]$, $\sigma^{\nu\rho} = \frac{i}{2}[\gamma^\nu, \gamma^\rho]$.

$\frac{i}{2}[\gamma^\mu, [\gamma^\nu, \gamma^\rho]] = \frac{i}{2}[\gamma^\mu, \gamma^\nu\gamma^\rho - \gamma^\rho\gamma^\nu] =$

$\frac{i}{2}(\gamma^\mu\gamma^\nu\gamma^\rho - \underline{\gamma^\nu\gamma^\rho\gamma^\mu} - \gamma^\mu\gamma^\rho\gamma^\nu + \underline{\gamma^\rho\gamma^\nu\gamma^\mu}) = (**)$

$\gamma^\nu\gamma^\rho\gamma^\mu = (2g^{\nu\rho} - \gamma^\rho\gamma^\nu)\gamma^\mu = 2g^{\nu\rho}\gamma^\mu - \gamma^\rho(2g^{\nu\mu} - \gamma^\mu\gamma^\nu) =$

$= 2g^{\nu\rho}\gamma^\mu - 2g^{\nu\mu}\gamma^\rho + \gamma^\rho\gamma^\mu\gamma^\nu$

$\gamma^\rho\gamma^\nu\gamma^\mu = \dots 2g^{\rho\nu}\gamma^\mu - 2g^{\rho\mu}\gamma^\nu + \gamma^\nu\gamma^\mu\gamma^\rho$

$(**) = \frac{i}{2}\Big(\gamma^\mu\underline{\gamma^\nu}\gamma^\rho - 2g^{\nu\rho}\gamma^\mu + 2g^{\nu\mu}\gamma^\rho - \underline{\gamma^\rho\gamma^\mu}\gamma^\nu$

$-\gamma^\mu\underline{\gamma^\rho}\gamma^\nu + 2g^{\rho\nu}\gamma^\mu - 2g^{\rho\mu}\gamma^\nu + \gamma^\nu\underline{\gamma^\mu}\gamma^\rho\Big)$

$= \frac{i}{2}\left(2g^{\mu\nu}\gamma^\rho + 2g^{\nu\mu}\gamma^\rho - 2g^{\mu\rho}\gamma^\nu - 2g^{\rho\mu}\gamma^\nu\right) =$

$= \underline{2i\left(g^{\mu\nu}\gamma^\rho - g^{\mu\rho}\gamma^\nu\right)}$

A 2] <u>Variante A</u> $1/2 \otimes 1/2 \otimes 1/2$

$\vec{J}_{12} = \vec{S}_1 + \vec{S}_2 \Rightarrow J_{12} = 0, 1;$

$\vec{J} = \vec{S}_3 + \vec{J}_{12} \Rightarrow J = 1/2;\ 3/2,\ 1/2$

$J = 3/2 \Rightarrow m_J = -3/2, -1/2, 1/2, 3/2 \Rightarrow$ 4 Zustände

$J = 1/2,\ J_{12} = 1 \Rightarrow m_J = -1/2, 1/2 \Rightarrow$ 2 Zustände

$J = 1/2,\ J_{12} = 0 \Rightarrow m_J = -1/2, 1/2 \Rightarrow$ 2 Zustände

} 8 Zustände

$\Rightarrow \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = (0 \oplus 1) \otimes \frac{1}{2} = (0 \times \frac{1}{2}) \oplus (1 \otimes \frac{1}{2}) = \frac{1}{2} \oplus (\frac{1}{2} \oplus \frac{3}{2})$

<u>Variante B</u> $1/2 \otimes 1/2 \otimes 1$

$\vec{J}_{12} = \vec{S}_1 + \vec{S}_2 \Rightarrow J_{12} = 0, 1.$

$\vec{J} = \vec{S}_3 + \vec{J}_{12} \Rightarrow J = 0, 1, 2$

$J = 2 \Rightarrow m_J = -2, -1, 0, 1, 2 \rightarrow$ 5 Zustände

$J = 1,\ J_{12} = 1 \Rightarrow m_J = -1, 0, 1 \Rightarrow$ 3 Zustände

$J = 0 \quad m_J = 0$ 1 Zustand

} 12 Zustände

$J = 1 \quad J_{12} = 0 \quad m_J = -1, 0, 1 \Rightarrow$ 3 Zustände

$\Rightarrow \left(\frac{1}{2} \otimes \frac{1}{2}\right) \otimes 1 = (0 \oplus 1) \otimes 1 = 1 \oplus (1 \otimes 1) = 1 \oplus (0 \oplus 1 \oplus 2)$

A·3] a) $S_{12} = 2\left[3\frac{(\vec S\cdot\vec r)^2}{r^2} - \vec S^2\right]$

$$\vec S\cdot\vec r = S_x x + S_y y + S_z z \qquad \left\{\begin{array}{l} S_x = \frac{S_+ + S_-}{2} \\ S_y = \frac{S_+ - S_-}{2i}\end{array}\right.$$

$$\vec r = r(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$$

$$\vec S\cdot\vec r = [(S_x\cos\phi + S_y\sin\phi)\sin\theta + S_z\cos\theta]\,r =$$

$$= \frac12\left[(S_+ e^{-i\phi} + S_- e^{i\phi})\sin\theta + S_z\cos\theta\right] r$$

$$\Rightarrow \frac{(\vec S\cdot\vec r)^2}{r^2} = \frac14\left[\overbrace{(S_+^2 e^{-2i\phi} + S_-^2 e^{2i\phi})\sin^2\theta}^{\sim Y_{2,\pm2}} + (S_+S_- + S_-S_+)\sin^2\theta\right]$$

$$+ S_z^2\cos^2\theta + \frac12(S_zS_+ + S_+S_z)\underbrace{e^{-i\phi}\sin\theta\cos\theta}_{\sim Y_{2,-1}} +$$

$$+ \frac12(S_zS_- + S_-S_z)\underbrace{e^{i\phi}\sin\theta\cos\theta}_{\sim Y_{2,1}}$$

$S_+S_- + S_-S_+ =$ (siehe Hinweise) $= 2(S_x^2 + S_y^2) = 2(\vec S^2 - S_z^2)$

$$\Rightarrow \frac14(S_+S_- + S_-S_+)\sin^2\theta + S_z^2\cos^2\theta = \frac12(\vec S^2 - S_z^2)(1-\cos^2\theta) + S_z^2\cos^2\theta$$

$$= \frac12\vec S^2(1-\cos^2\theta) + S_z^2\left(\frac{3\cos^2\theta - 1}{2}\right)$$

$$\Rightarrow S_{12} = \frac32(S_+^2 e^{-2i\phi}\sin^2\theta + S_-^2 e^{2i\phi}\sin^2\theta) + 3(S_zS_+ + S_+S_z)e^{-i\phi}\sin\theta\cos\theta +$$

$$+ 3(S_-S_z + S_zS_-)e^{i\phi}\sin\theta\cos\theta + 3S_z^2(3\cos^2\theta - 1)$$

$$- \vec S^2(3\cos^2\theta - 1) =$$

$$= \sqrt{\frac{24\pi}{5}}\Big[S_+^2 Y_{2,-2} + S_-^2 Y_{22} + (S_+S_z + S_zS_+)Y_{2,-1} - (S_zS_- + S_-S_z)Y_{21}$$

$$+ \sqrt{\frac23}(3S_z^2 - \vec S^2)Y_{20}\Big]$$

b) $T_{-2}^{(2)} = Y_{2,-2}$, $S_{+2}^{(2)} = S_+^2$; $T_{-1}^{(2)} = Y_{2,-1}$, $S_1^{(2)} = -(S_+S_z + S_zS_+)$

$T_{+2}^{(2)} = Y_{2,2}$, $S_{-2}^{(2)} = S_-^2$; $T_1^{(2)} = Y_{2,1}$, $S_{-1}^{(2)} = S_-S_z + S_zS_-$

$T_0^{(2)} = Y_{20}$; $S_0^{(2)} = \sqrt{\frac23}(3S_z^2 - \vec S^2)$

$[L_z, Y_{lm}] = m\hbar Y_{lm}$, $[L_\pm, Y_{lm}] = \hbar\sqrt{l(l+1) - m(m\pm1)}\,Y_{l,m\pm1}$

$[\vec L, \vec S] = 0 \Rightarrow [J_z, Y_{lm}] = m\hbar Y_{lm}, \ldots$

$\Rightarrow Y_{lm}$ = irr. Tensor l-te Stufe.

noch zu zeigen: $S_m^{(2)}$ = irr. Tensor 2-te Stufe.

$$(\Leftrightarrow)\quad [S_z, S_m^{(2)}] = \hbar m S_m^{(2)}$$

$$[S_\pm, S_m^{(2)}] = \hbar \sqrt{2\cdot 3 - m(m\pm 1)}\, S^{(2)}_{m\pm 1}.$$

$$[S_z, S^{(2)}_{\pm 2}] = [S_z, S_\pm^2] = [S_z, S_\pm]S_\pm + S_\pm[S_z, S_\pm] = \pm 2\hbar (S_\pm)^2 = \pm 2\hbar S^{(2)}_{\pm 2} \checkmark$$

$$[S_z, S^{(2)}_{\pm 1}] = \mp[S_z, S_z S_\pm + S_\pm S_z] = \mp(S_z[S_z, S_\pm] + [S_z, S_\pm]S_z) =$$
$$= (\mp) \pm \hbar (S_z S_\pm + S_\pm S_z) = \pm \hbar S^{(2)}_{\pm 1} \checkmark$$

$$[S_z, S_0^{(2)}] = \sqrt{\tfrac{2}{3}}\,[S_z, 3S_z^2 - \vec{S}^2] = 0 \checkmark$$

$$[S_+, S^{(2)}_{+2}] = [S_+, (S_+)^2] = 0 \checkmark$$

$$[S_+, S^{(2)}_{-2}] = [S_+, (S_-)^2] = S_-[S_+, S_-] + [S_+, S_-]S_- = 2\hbar\,[\underbrace{S_- S_z + S_z S_-}_{S^{(2)}_{-1}}]$$
$$= 2\hbar\, S^{(2)}_{-1} \checkmark$$

$$[S_+, S^{(2)}_{-1}] = [S_+, S_- S_z + S_z S_-] = [S_+, S_-]S_z + S_-[S_+, S_z] + S_z[S_+, S_-] +$$
$$+ [S_+, S_z]S_- = +2\hbar S_z^2 + S_-(-\hbar S_+) + S_z \cdot 2\hbar S_z + (-\hbar S_+)S_-$$
$$= 4\hbar S_z^2 - \hbar(S_+ S_- + S_- S_+) = \hbar(4S_z^2 - 2\vec{S}^2 + 2S_z^2) =$$
$$= 2\hbar(3S_z^2 - \vec{S}^2) = \hbar\sqrt{6}\, S_0^{(2)} \checkmark$$

$$[S_+, S_0^{(2)}] = \sqrt{\tfrac{2}{3}}\,[S_+, 3S_z^2 - \vec{S}^2] = \sqrt{\tfrac{2}{3}}\,[S_+, 3S_z^2] = \sqrt{6}\, S_z[S_+, S_z] +$$
$$+ \sqrt{6}\,[S_+, S_z]S_z = \hbar\sqrt{6}\,(-S_z S_+ - S_+ S_z) =$$
$$= +\hbar\sqrt{6}\, S^{(2)}_{+1}$$

$$[S_+, S_1^{(2)}] = -[S_+, S_+ S_z + S_z S_+] = -(S_+[S_+, S_z] + [S_+, S_z]S_+) =$$
$$= +\hbar S_+ S_+ \cdot 2 = +2\hbar S_+^2 = +2\hbar S^{(2)}_{+2}$$

Der Rest ($[S_-, S_m^{(2)}] = \dots$) ganz analog

$= 4\hbar S_z^2 - \hbar (S_- S_+ + S_+ S_-) = 4\hbar S_z^2 - 2\hbar(\vec{S}^2 - S_z^2) = 6\hbar S_z^2 - 2\hbar \vec{S}^2 =$

$= 2\hbar (3S_z^2 - \vec{S}^2) = \hbar\sqrt{6}\, S_0^{(2)}$ ✓

A.4] $H = H_0 + H_{LS} + H_{HF}$

a) $H_{LS} = \xi(r)\, \vec{L}\cdot\vec{S} = \xi(r) \dfrac{(\vec{J}^2 - \vec{S}^2 - \vec{L}^2)}{2}$

$H_{HF} = \eta(r)\, \vec{S}_1 \cdot \vec{S}_2 = \eta(r) \dfrac{(\vec{S}^2 - \vec{S}_1^2 - \vec{S}_2^2)}{2}$

$[\vec{J}, \vec{J}^2] = 0$;

$[\vec{J}, \vec{S}^2] = [\vec{J}, \vec{L}^2] = 0$ da $[\vec{S}, \vec{S}^2] = [\vec{L}, \vec{L}^2] = 0$

$[\vec{J}, \vec{S}_1^2] = [\vec{J}, \vec{S}_2^2] = 0$ da $[\vec{S}_i, \vec{S}_i^2] = 0$

(Siehe auch A 1, Blatt 2)

b) $\{ \vec{J}^2, J_z ; \vec{L}^2, \vec{S}^2, \vec{S}_1^2, \vec{S}_2^2 \}$

c) $\vec{S} = \vec{S}_1 + \vec{S}_2$ $\quad S = 0 ; 1.$

$l = 0,\ S = 0 \Rightarrow J = 0$ keine Aufspaltung

$l = 0,\ S = 1 \Rightarrow J = 1$ " —

$l = 1,\ S = 0 \Rightarrow J = 1$ " —

$l = 1,\ S = 1 \Rightarrow J = 2, 1, 0. \Rightarrow$ 3-fache Aufspaltung

c) $\langle n\ l = 1\ s = 1\ J\ J_z | H | \dots \rangle = H_0(n, e)$

$+ \dfrac{\xi(r)}{2} \left(j(j+1) - s(s+1) - e(e+1) \right) \hbar^2 + \dfrac{\eta(r)}{2} \left(s(s+1) - \dfrac{3}{4} \cdot 2 \hbar^2 \right)$

$\Rightarrow \Delta E_{j, j+1} = \dfrac{\xi(r)}{2} \left[(j+2)(j+1) - (j+1) j \right] = \dfrac{\xi(r)}{2} (j+1) \cdot 2$

$\Rightarrow l = 1, S = 1$ ———— $j = 2$ $\quad \Delta E_{j=1, j=2} = \xi(r) \cdot 2$

———— $j = 1 \Rightarrow \Delta E_{j=0, j=1} = \xi(r)$

———— $j = 0$

A.4 / Variante B : $p^\mu \to i\partial^\mu$

Minimale Kopplung $\Rightarrow p^\mu \to p^\mu - \varepsilon A^\mu$

Eichtrans. in kovarianter Form

$A^\mu \to A^\mu - \partial^\mu \Delta(x)$ wobei $\partial^\mu = \begin{pmatrix} \frac{1}{c}\frac{\partial}{\partial t} \\ -\vec{\nabla} \end{pmatrix} \overset{\text{"}c=1\text{"}}{=} \begin{pmatrix} \partial_t \\ -\vec{\nabla} \end{pmatrix}$

$\psi(x) \to \psi'(x) = \psi(x)\, e^{i\varepsilon \cdot \Delta(x)}$

Zu zeigen: $(i\partial_\mu - \varepsilon A'_\mu)(i\partial^\mu - \varepsilon A'^\mu)\psi'(x) = m^2 \psi'(x)$

oder $\Leftrightarrow [(\partial_\mu + i\varepsilon A'_\mu)(\partial^\mu + i\varepsilon A'^\mu) + m^2]\psi'(x) = 0$.

$$\underline{(\partial^\mu + i\varepsilon A'^\mu)\psi(x)e^{i\varepsilon\Delta(x)}} = e^{i\varepsilon\Delta(x)}[(\partial^\mu \Delta)(i\varepsilon) + \partial^\mu + i\varepsilon A^\mu - i\varepsilon(\partial^\mu\Delta)]\psi(x)$$
$$= e^{i\varepsilon\Delta(x)}[\partial^\mu + i\varepsilon A^\mu]\psi(x)$$

$$\underline{(\partial_\mu + i\varepsilon A'_\mu)[e^{i\varepsilon\Delta(x)}(\partial^\mu + i\varepsilon A^\mu)\psi(x)]} =$$
$$= e^{i\varepsilon\Delta(x)}[(\partial_\mu \Delta(x))(i\varepsilon) + \partial_\mu + i\varepsilon A_\mu - i\varepsilon\partial_\mu\Delta][(\partial^\mu + i\varepsilon A^\mu)\psi]$$
$$= e^{i\varepsilon\Delta(x)}[\underbrace{(\partial_\mu + i\varepsilon A_\mu)(\partial^\mu + i\varepsilon A^\mu)\psi(x)}_{m^2\psi(x)}] = m\psi'(x)$$

$$\Rightarrow \underline{(\partial_\mu + i\varepsilon A'_\mu)(\partial^\mu + i\varepsilon A'^\mu)\psi'(x) = m^2\psi'(x)}$$