Moderne Physik - Klausur 1

Musterlösung

Aufgabe 1:

(i)

$$\mathrm{tr}(\not p_1 \not p_2 \not p_2 \not p_1) = 4\,(p_1^2)(p_2^2)$$

$$\mathrm{tr}(\not p_1 \not p_2 \not p_1 \not p_2) = 8\,(p_1 \cdot p_2)^2 - 4\,(p_1^2)(p_2^2)$$

(ii)

$$k'^{\mu} = \Lambda^{\mu}{}_{\nu}\, k^{\nu}$$

(iii)

$$\bar\psi\,(i\overleftarrow{\not\partial} + e\not A + m) = 0$$

(iv)

$$P|\psi\rangle = |\psi\rangle \quad \text{für Grundzustand}$$

$$P x = -x P \,, \quad P^{+}P = P^2 = \mathbb{1}$$

$$\begin{aligned}
\langle\psi|x|\psi\rangle &= \langle\psi|P^{+}P\,x|\psi\rangle \\
&= -\langle\psi|P^{+}\,x\,P|\psi\rangle \\
&= -\langle\psi|x|\psi\rangle \\
&= 0
\end{aligned}$$

(v)

$$\gamma^1\gamma^2 + \gamma^2\gamma^1 = \{\gamma^1, \gamma^2\} = 2g^{12} = 0$$

Aufgabe 2:

(i)

$$P_{12} = \frac{1}{\hbar^2} \left| \int_0^\infty dt' \; e^{i(E_2 - E_1)\cdot t'/\hbar} \; \langle 2 | V(t') | 1 \rangle \right|^2$$

mit $V(t) = e \cdot E(t) \cdot z = e E_0 \, e^{-t/\tau} \cdot z$

$$E_n = \hbar\omega(n + 1/2)$$

$$\Rightarrow \quad P_{12} = \frac{e^2 E_0^2}{\hbar^2} \underbrace{\left| \langle 2 | z | 1 \rangle \right|^2}_{= \frac{\hbar}{m\omega}} \cdot \underbrace{\left| \int_0^\infty dt' \; e^{i\omega t'} \; e^{-t'/\tau} \right|^2}_{= \frac{\tau^2}{1 + \omega^2\tau^2}}$$

$$= \frac{e^2 E_0^2}{\hbar m \omega} \cdot \frac{\tau^2}{1 + \omega^2 \tau^2}$$

(ii)

$$P_{13} = \frac{1}{\hbar^2} \left| \int_0^\infty dt' \; e^{i(E_3 - E_1) t'/\hbar} \; e^{-t'/\tau} \; \underbrace{\langle 3 | z | 1 \rangle}_{= 0} \right|^2$$

$$= 0$$

Aufgabe 3:

(i)

U unitär $\Rightarrow$ $UU^+ = 1$

$\beta^+ = \beta$; $\alpha^+ = \alpha$ $\Rightarrow$ $U^+ = U$

$$UU^+ = U^2 = \left(C\cdot\beta + \frac{\vec{\alpha}\cdot\vec{p}}{2m} \right)^2$$

$$= C^2\beta^2 + \frac{C}{2m}\underbrace{\{\beta, \vec{\alpha}\}}_{=0}\cdot\vec{p} + \underbrace{\frac{(\vec{\alpha}\cdot\vec{p})^2}{4m^2}}_{=\frac{\vec{p}^2}{4m^2}}$$

$$= C^2 + \frac{\vec{p}^2}{4m^2} = \underline{\underline{1}}$$

(ii)

$$U = \sqrt{1-\frac{\vec{p}^2}{4m^2}}\,\beta + \frac{\vec{\alpha}\cdot\vec{p}}{2m} = \left(1 - \frac{\vec{p}^2}{8m^2}\right)\beta + \frac{\vec{\alpha}\cdot\vec{p}}{2m} + \mathcal{O}\left(\frac{1}{m^3}\right)$$

(iii)

$$\tilde{H}_m = U\cdot m(\beta-1)\,U$$

$$= \left(m - \frac{\vec{p}^2}{4m^2}\right)(\beta-1) + \frac{\vec{\alpha}\cdot\vec{p}}{2m}\, m(\beta-1)\,\frac{\vec{\alpha}\cdot\vec{p}}{2m}$$

$$+ \frac{1}{2}\left[\beta(\beta-1)\,\vec{\alpha}\cdot\vec{p} + \vec{\alpha}\cdot\vec{p}\,(\beta-1)\beta\right]$$

nicht-diagonal $\hat{=}$ ungerade Anzahl von α's :

$$\tilde{H}_m^{nd} = \frac{1}{2}\left[\beta(\beta-1)\,\vec{\alpha}\cdot\vec{p} + \vec{\alpha}\cdot\vec{p}\,(\beta-1)\beta\right]$$

$$= \vec{\alpha}\cdot\vec{p}$$

(iv)

$$\frac{1}{2m}\left[\vec{\sigma}\cdot(\vec{p}-e\vec{A})(\vec{\sigma}\cdot\vec{p}) + (\vec{\sigma}\cdot\vec{p})\cdot\vec{\sigma}\cdot(\vec{p}-e\vec{A}) - \vec{p}^{\,2}\right]$$

$$= \frac{1}{2m}\left[2\,(\vec{\sigma}\cdot\vec{p})^2 - \vec{p}^{\,2} - e\left((\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{p}) + (\vec{\sigma}\cdot\vec{p})(\vec{\sigma}\cdot\vec{A})\right)\right)$$

$$\overset{(*)}{=} \frac{1}{2m}\left[\vec{p}^{\,2} - e\left\{\vec{A}\cdot\vec{p} + \vec{p}\cdot\vec{A}\right\} - e\cdot i\,\vec{\sigma}\cdot(\vec{p}\times\vec{A})\right]$$

$$\overset{(**)}{=} \frac{1}{2m}\left[\vec{p}^{\,2} - e\left\{\vec{A}\cdot\vec{p} + \vec{p}\cdot\vec{A}\right\} - 2\,e\,\vec{S}\cdot\vec{B}\right]$$

$g_e = 2$

(*)

$$(\vec{\sigma}\cdot\vec{p})^2 = \vec{p}^{\,2}$$

$$(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{p}) = \vec{A}\cdot\vec{p} + i\,\vec{\sigma}\cdot(\vec{A}\times\vec{p})$$

$$(\vec{\sigma}\cdot\vec{p})(\vec{\sigma}\cdot\vec{A}) = \vec{p}\cdot\vec{A} + i\,\vec{\sigma}\;\vec{p}\times\vec{A} = \vec{p}\cdot\vec{A} + i\,\vec{\sigma}\cdot(\vec{p}\times\vec{A}) - i\,\vec{\sigma}\,(\vec{A}\times\vec{p})$$

(**)

$$(\vec{p}\times\vec{A}) = \frac{\hbar}{i}\,(\vec{\nabla}\times\vec{A}) = \frac{\hbar}{i}\,\vec{B} \quad ; \quad \vec{\sigma} = \frac{2}{\hbar}\,\vec{S}$$