Aufgabe 21

a) $S = -\dfrac{\partial \Omega}{\partial T}$

$$\Omega = -kT \sum_\lambda{}' \ln\left[1 + e^{-(\varepsilon_\lambda - \mu)/kT}\right] \quad \text{Fermionen}$$

$$\Omega = kT \sum_\lambda{}' \ln\left[1 - e^{-(\varepsilon_\lambda - \mu)/kT}\right] \quad \text{Bosonen}$$

1. Fermionen

$$-\frac{\partial \Omega}{\partial T} = k \sum_\lambda{}' \ln\left[1 + e^{-(\varepsilon_\lambda - \mu)/kT}\right] + k\cancel{T} \sum_\lambda{}' \left[\frac{e^{-(\varepsilon_\lambda-\mu)/kT}\left(\frac{\varepsilon_\lambda - \mu}{kT}\right)\frac{1}{\cancel{T}}}{1 + e^{-(\varepsilon_\lambda-\mu)/kT}}\right]$$

$$\underset{\uparrow}{=} k \sum_\lambda{}' \ln\left[1 + e^{-\alpha_\lambda}\right] + k \sum_\lambda{}' \frac{\alpha_\lambda e^{-\alpha_\lambda}}{1 + e^{-\alpha_\lambda}}$$

$\alpha_\lambda := \dfrac{\varepsilon_\lambda - \mu}{kT}$

$$\Rightarrow S = -\frac{\partial \Omega}{\partial T} = k \sum_\lambda{}' \ln\left[1 + e^{-\alpha_\lambda}\right] + k \sum_\lambda{}' \frac{\alpha_\lambda e^{-\alpha_\lambda}}{1 + e^{-\alpha_\lambda}}$$

angegebene Beziehung für S (zu zeigen)

$$S \overset{?}{=} -k \sum_\lambda{}' \left[f_\lambda \ln f_\lambda + (1 - f_\lambda)\ln(1 - f_\lambda)\right] \quad \text{mit } f_\lambda = \frac{1}{e^{\alpha_\lambda} + 1}$$

$$= -k \sum_\lambda{}' \left[\frac{1}{e^{\alpha_\lambda}+1} \ln\left(\frac{1}{e^{\alpha_\lambda}+1}\right) + \frac{e^{\alpha_\lambda}}{e^{\alpha_\lambda}+1} \ln\left(\frac{e^{\alpha_\lambda}}{e^{\alpha_\lambda}+1}\right)\right]$$

$$= -k \sum{}' \left[\left(\frac{1}{e^{\alpha_\lambda}+1} + \frac{e^{\alpha_\lambda}}{e^{\alpha_\lambda}+1}\right) \ln\left(\frac{1}{e^{\alpha_\lambda}+1}\right) + \frac{\alpha_\lambda e^{\alpha_\lambda}}{\ldots 1}\right]$$

$$= -\vartheta \sum_\lambda{}' \left[\ln \frac{e^{-\alpha_\lambda}}{1+e^{-\alpha_\lambda}} + \frac{\alpha_\lambda}{1+e^{-\alpha_\lambda}} \right]$$

$$\Rightarrow \quad -\vartheta \sum_\lambda{}' \left[f_\lambda \ln f_\lambda + (1-f_\lambda) \ln (1-f_\lambda) \right]$$

$$= \vartheta \sum_\lambda{}' \ln \left[1+e^{-\alpha_\lambda} \right] - \vartheta \sum_\lambda{}' \underbrace{\left[-\alpha_\lambda + \frac{\alpha_\lambda}{1+e^{-\alpha_\lambda}} \right]}_{= -\frac{\alpha_\lambda e^{-\alpha_\lambda}}{1+e^{-\alpha_\lambda}}}$$

$$= -\frac{\partial \Omega}{\partial T} \qquad \square$$

2. Bosonen

$$-\frac{\partial \Omega}{\partial T} = -\vartheta \sum_\lambda{}' \ln \left[1 - e^{-(\varepsilon_\lambda - \mu)/\vartheta T} \right]$$

$$- \vartheta \cancel{T} \sum_\lambda{}' \left[\frac{e^{-(\varepsilon_\lambda-\mu)/\vartheta T} \left(\frac{\varepsilon_\lambda - \mu}{\vartheta T} \right) \cancel{\frac{1}{T}}}{1 - e^{-(\varepsilon_\mu - \mu)/\vartheta T}} \right]$$

$$= -\vartheta \sum_\lambda{}' \ln \left[1 - e^{-\alpha_\lambda} \right] + \vartheta \sum_\lambda{}' \frac{\alpha_\lambda e^{-\alpha_\lambda}}{1-e^{-\alpha_\lambda}}$$

mionen

$$\frac{\delta \Omega}{\delta f_\lambda} = (\varepsilon_\lambda - \mu) + kT\left[\ln f_\lambda \cancel{+1} - \ln(1-f_\lambda) \cancel{-1}\right]$$

$$\stackrel{!}{=} 0$$

$$\Rightarrow \ln\frac{f_\lambda}{1-f_\lambda} = -\alpha_\lambda$$

$$f_\lambda = e^{-\alpha_\lambda}(1-f_\lambda)$$

$$f_\lambda = \frac{1}{e^{\alpha_\lambda}+1} \qquad \boxed{1/2}$$

osonen

$$\frac{\delta \Omega}{\delta g_\lambda} = (\varepsilon_\lambda - \mu) + kT \ln\frac{g_\lambda}{1+g_\lambda} \stackrel{!}{=} 0$$

$$\Rightarrow g_\lambda = \frac{1}{e^{\alpha_\lambda}-1} \qquad \boxed{1/2}$$

Aufgabe 3

$$H = -J\sigma_1\sigma_2 - \mu B(\sigma_1 + \sigma_2)\,; \quad J>0\,, \quad \sigma_{1/2} = \pm 1/2$$

) $B=0$

Zustand	Energie
$\uparrow\uparrow$	$-J/4$
$\downarrow\uparrow$	$J/4$
$\uparrow\downarrow$	$J/4$
$\downarrow\downarrow$	$-J/4$

$\boxed{1/2}$

$$Z_2 = 2\left(e^{\beta J/4} + e^{-\beta J/4}\right) = 4\cosh(\beta J/4) \qquad \boxed{1/2}$$

$$U = -\frac{1}{Z_2}\partial_\beta Z_2 = -\frac{J}{4}\tanh(\beta J/4) \qquad \boxed{1/2}$$

$$C_V = \frac{\partial U}{\partial T} = k\left(\frac{J}{4kT}\right)^2 \frac{1}{\cosh^2(J/4kT)} \qquad \boxed{1/2}$$

b) $B \neq 0$

Zustand	Energie
$\uparrow\uparrow$	$-J/4 - \mu B$
$\uparrow\downarrow$	$J/4$
$\downarrow\uparrow$	$J/4$
$\downarrow\downarrow$	$-J/4 + \mu B$

$\boxed{1/2}$

$$Z_2 = 2\,e^{-\beta J/4} + e^{+\beta J/4}\left(e^{\beta\mu B} + e^{-\beta\mu B}\right)$$

$$= 2\left[e^{-\beta J/4} + e^{\beta J/4}\cosh(\beta\mu B)\right]$$

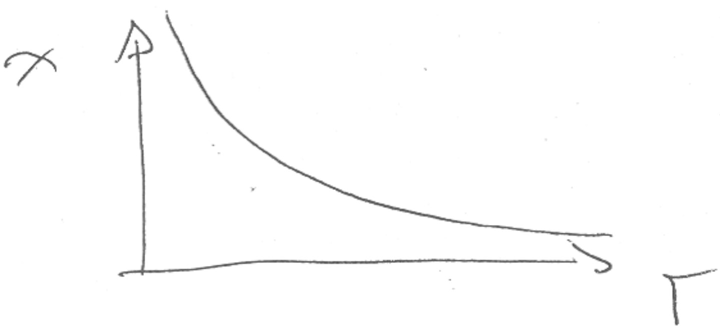
$$F = -kT \ln Z_2$$

$$M(T) = -\left(\frac{\partial F}{\partial B}\right)_T = \frac{1}{\cancel{\beta}}\,\frac{\cancel{\beta}\mu\, e^{\beta J/4}\sinh(\beta\mu B)}{e^{-\beta J/4} + e^{\beta J/4}\cosh(\beta\mu B)} \qquad \boxed{1/2}$$

$$\chi(T) = \left.\frac{\partial M}{\partial B}\right|_{B=0} = \mu\, e^{\beta J/4}\,\frac{2\beta\mu\cosh(\beta J/4)}{4\cosh^2(\beta J/4)}$$

$$= \frac{1}{2}\,\frac{\mu^2}{kT}\,\frac{e^{\beta J/4}}{\cosh(\beta J/4)} = \frac{\mu^2}{kT}\,\frac{1}{1+e^{-J/2kT}} \qquad \boxed{1/2}$$

c) $$\chi(T) \cong \begin{cases} \dfrac{\mu^2}{kT} & T \to 0 \\[2ex] \dfrac{1}{2}\dfrac{\mu^2}{kT} & T \to \infty \end{cases}$$



$\boxed{1}$

Aufgabe 4

a) $\hat{M} = \mu \sum_{i=1}^{N} \sigma_i$ $\boxed{1/2}$

b) σ_i : random variable (Zufallsgröße) mit Werte $\pm\frac{1}{2}$

$\Rightarrow$ $\hat{M}$ ist Zufallsgröße

$\Rightarrow$ für $N \to \infty$ die Wahrscheinlichkeitsverteilung ist Gauß-verteilung

$\Rightarrow$ $\frac{\langle \Delta \hat{M} \rangle}{\langle \hat{M} \rangle} \sim \frac{1}{\sqrt{N}} \xrightarrow{N \to \infty} 0$

$\Rightarrow$ $\langle \hat{M} \rangle = \hat{M}(T,B) \underset{N \to \infty}{=} \hat{M}$ $\boxed{1/2}$

$\Rightarrow$ $H = \underbrace{\left(-\frac{J}{N}\frac{M}{M} - \mu B\right)}_{= -\mu B_{eff}} \sum_{i=1}^{N} \sigma_i$ $\boxed{1/2}$

c) $Z = \left(2\cosh\left(\frac{\mu B_{eff}}{2 k T}\right)\right)^N$ $\boxed{1/2}$

$M = \frac{\partial \ln Z}{\partial(\beta B_{eff})} = N\frac{\mu}{2} \tanh\left(\frac{\mu B_{eff}}{2 k T}\right)$ $\boxed{1}$

d) $B = 0 \Rightarrow B_{eff} = + \frac{J}{N} \frac{M}{\mu^2}$ $\left(\Rightarrow M = \mu B_{eff} \frac{\mu N}{J} \right.$

$\Rightarrow M = x \frac{2 k T \mu N}{J} = N \frac{\mu}{2} \tanh(x)$.

$\Rightarrow \quad x \frac{4 k T}{J} = \tanh(x) \quad \Big| \partial_x \cdot$ und $x = 0$

$\Rightarrow T_c : \frac{4 k T_c}{J} = \frac{1}{\cosh^2(x)} \Big|_{x=0} = 1$

$\Rightarrow T_c = \frac{J}{4k}$ $\boxed{1}$

e) $T \approx T_c$ $\quad \tanh x \simeq x - \frac{1}{3} x^3$

$x \frac{T}{T_c} = x - \frac{1}{3} x^3$

$\Rightarrow x \left(1 + \frac{T - T_c}{T_c} \right) = x - \frac{1}{3} x^3$

$\Rightarrow x = \left(3 \frac{T_c - T}{T_c} \right)^{1/2}$

$M(T) = \frac{N\mu}{J} 2 k T_c \sqrt{3} \sqrt{\frac{T - T_c}{T_c}}$ $\boxed{1}$

ufgabe 5

(a) $|L\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ $\qquad |R\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

$$\Rightarrow H = \begin{pmatrix} \varepsilon - \Delta/2 & \tau \\ \tau & \varepsilon + \Delta/2 \end{pmatrix}$$

$$H \underset{\Delta=0}{=} \begin{pmatrix} \varepsilon & \tau \\ \tau & \varepsilon \end{pmatrix}$$

Eigenwerte:
$$(\varepsilon-\lambda)^2 - \tau^2 = 0$$

$$\Rightarrow \lambda_{1/2} = \varepsilon \pm \tau \qquad \boxed{1/2}$$

Eigenzustände:
$$e_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$
$$e_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \qquad \boxed{1/2}$$

b) $|\psi(0)\rangle = \begin{pmatrix}1\\0\end{pmatrix} = c_1 e_1 + c_2 e_2$

$$\Rightarrow c_1 = c_2 = \frac{1}{\sqrt{2}}$$

$$\Rightarrow |\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{i(\varepsilon+\tau)/\hbar\, t} e_1 + e^{i(\varepsilon-\tau)/\hbar\, t} e_2 \right) \qquad \boxed{1/2}$$

$$P_L = |\langle L|\psi(t)\rangle|^2$$
$$= \left| \frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix} \left(e^{i(\varepsilon+\tau)/\hbar\, t}\begin{pmatrix}1\\1\end{pmatrix} + e^{i(\varepsilon-\tau)/\hbar\, t}\begin{pmatrix}1\\-1\end{pmatrix} \right) \right|$$
$$= \left| \frac{1}{2} e^{i\varepsilon/\hbar\, t}\, 2\cos \tau/\hbar\, t \right|^2 = \cos^2(\tau/\hbar\, t) \qquad \boxed{1/2}$$

(11)

c) reine Zustände: $\hat{\omega}_1$, $\hat{\omega}_2$, $\hat{\omega}_4$

über Entropie $= 0$

oder $\hat{\omega}^2 = \hat{\omega}$

$\boxed{1/2}$

gemischte Zustände: ω_3

Eigenzustände: $\hat{\omega}_2 = \frac{1}{\sqrt{2}}(1\ 1)\ \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

$\Rightarrow$ entspricht $|e_1\rangle\langle e_1|$

[An]fangszustand: $\hat{\omega}_1 = (1\,0)\begin{pmatrix}1\\0\end{pmatrix}$

$\boxed{1/2}$

$\Rightarrow$ entspricht $|L\rangle\langle L|$

d) $$U = e^{-\hat{H}/\hbar \cdot t}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-i t}{\hbar}\right)^n \underbrace{\begin{pmatrix}\varepsilon & \tau\\ \tau & \varepsilon\end{pmatrix}^n}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}(\varepsilon+\tau)^n & 0\\ 0 & (\varepsilon-\tau)^n\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}e^{-i(\varepsilon+\tau)/\hbar \cdot t} & 0\\ 0 & e^{-i(\varepsilon-\tau)/\hbar \cdot t}\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1 & -1\end{pmatrix} e^{-\frac{i\varepsilon}{\hbar}t}\begin{pmatrix}e^{-i\tau/\hbar t} & e^{-i\tau/\hbar t}\end{pmatrix}$$

$$\Rightarrow \mathcal{U} = \frac{1}{2}\, e^{-i\frac{\pi}{5}t} \begin{pmatrix} 2\cos\frac{\pi}{5}t & -2i\sin\frac{\pi}{5}t \\ -2i\sin\frac{\pi}{5}t & 2\cos\frac{\pi}{5}t \end{pmatrix}$$

$$= e^{-i\frac{\pi}{5}t} \begin{pmatrix} \cos\frac{\pi}{5}t & i\sin\frac{\pi}{5}t \\ -i\sin\frac{\pi}{5}t & \cos\frac{\pi}{5}t \end{pmatrix}$$

$\boxed{1/2}$

$$\hat{w}(t) = \hat{U}\, \underbrace{\hat{w}(0)}_{=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}\, \hat{U}^{+}$$

Argumente je $\frac{\pi}{5}t$

$$= \begin{pmatrix} \cos & -i\sin \\ -i\sin & \cos \end{pmatrix} \begin{pmatrix} \cos & i\sin \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2 & i\sin\cos \\ -i\sin\cos & \sin^2 \end{pmatrix}$$

$\boxed{1/2}$

Observable: Teilchen links

Projektor auf $|L\rangle$

$\boxed{1/2}$

$$\Rightarrow \hat{O}_L = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \langle \hat{O}_L \rangle(t) = \mathrm{Tr}\{\hat{w}(t)\, \hat{O}_L\}$$

$$= \cos^2\left(\frac{\pi}{5}t\right)$$

$\boxed{1/2}$