

Moderne Theoretische Physik III SS 2014

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Klausur, 100 Punkte

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1. Kurzfragen

(10 + 10 + 10 + 10 + 10 = 50 Punkte)

- (a) Photons – density of states in
- $d = 3$
- . We have that
- $\epsilon(k) = ck$
- . Then

$$g(\epsilon)d\epsilon = g(k)dk = \frac{4\pi k^2}{(2\pi)^3} dk \quad (1)$$

and it follows that

$$g(\epsilon) = \frac{4\pi k^2}{(2\pi)^3} \left(\frac{d\epsilon}{dk} \right)^{-1} \quad (2)$$

using that $\frac{d\epsilon}{dk} = c$ we get

$$g(\epsilon) = \frac{1}{2(c\pi)^3} \epsilon^2 \quad (3)$$

The distribution function

$$n(p) = \frac{1}{e^{cp/T} - 1} \quad (4)$$

- (b) Equipartition theorem: Every quadratic degree of freedom in the Hamiltonian contributes
- $kT/2$
- to the internal energy. For a molecule of two atoms in
- $d = 3$
- , there is a vibrational mode that contributes
- $2\frac{kT}{2}$
- to the internal energy and
- $3\frac{kT}{2}$
- from the translation. Rotational degrees of freedom contribute
- $2\frac{kT}{2}$
- to the internal energy. Therefore
- $U = \frac{7}{2}NkT$
- , and hence
- $c = \frac{dU}{dT} = \frac{7}{2}Nk$
- .

- (c) Generally,
- n
- th order phase transition has the discontinuity in the
- n
- th derivative of the free energy. Entropy in first order transition: Since
- $S = -\frac{\partial F}{\partial T}$
- (first derivative of free energy), there will be a jump in entropy at first order transition, for example, see Fig 1. Also,
- $dS = \frac{dQ}{T}$
- and in a first order transition there is latent heat released so
- dS
- will be finite and hence the jump in entropy.

Entropy and specific heat in a second order phase transition. In a second order phase transition there is no latent heat and $dS = 0$ – entropy (which is a first derivative of free energy) is continuous at $T = T_c$, but there is a cusp in entropy since the second derivative of the free energy $c = T \frac{\partial S}{\partial T}$ is discontinuous, for details see Fig 2 and Fig 3.

- (d) We are given
- $\delta X = (A \ln V + B \ln T + C)\delta T + D \frac{T}{V} \delta V$
- . In order for this to be a thermodynamic quantity the second derivatives must commute i.e. we must have that
- $\frac{\partial^2 X}{\partial T \partial V} = \frac{\partial^2 X}{\partial V \partial T}$
- . After performing the partial derivatives this boils down to

$$\frac{\partial}{\partial V}(A \ln V + B \ln T + C) = \frac{\partial}{\partial T}(DT/V) \quad (5)$$

which is when $A = D$.

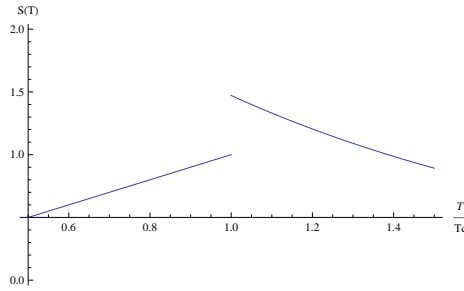


Abbildung 1: Entropy profile in the first order phase transition

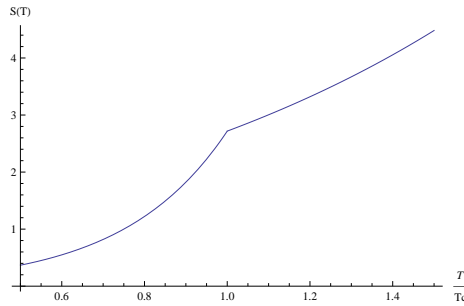


Abbildung 2: Entropy in a 2nd order phase transition

- (e) Two level system. The partition function is $Z = 1 + e^{-\frac{E_1}{kT}}$. The probability to be in the state of energy E_1 is given by

$$p_1 = \frac{e^{-\frac{E_1}{kT}}}{Z} = \frac{e^{-\frac{E_1}{kT}}}{1 + e^{-\frac{E_1}{kT}}} \quad (6)$$

If we put $p_1 = 1/4$, and substitute $x = e^{-\frac{E_1}{kT}}$, we get from the above that $x = 1/3$. Then taking the log one gets that $T = \frac{E_1}{k \ln 3}$.

2. Quadratic dispersion in $d = 2$ (5 + 5 + 5 + 10 = 25 Punkte)

- (a) Fermi energy. first we find the density of states $g(\epsilon)$. We have that $g(\epsilon)d\epsilon = g_s g(k)dk = \frac{2\pi k}{(2\pi)^3} dk$, where $g_s = 2$ is the spin factor. After using that $\frac{d\epsilon}{dk} = k/m$, one gets $g(\epsilon) = m/\pi$.

The number of particles is given by

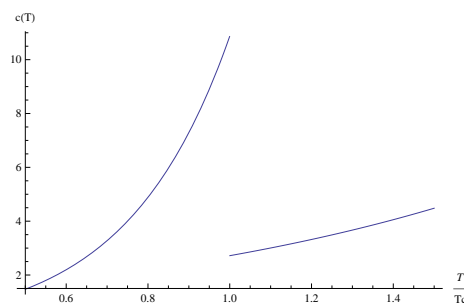


Abbildung 3: Specific heat in a 2nd order phase transition

$$N/V = \int_0^{\epsilon_F} g(\epsilon) d\epsilon = \frac{m}{\pi} \epsilon_F \quad (7)$$

therefore the Fermi energy is $\epsilon_F = \frac{\pi N}{m}$.

(b) The internal energy at $T = 0$. This is given by

$$\begin{aligned} E_{T=0}/V &= \int_0^{\epsilon_F} g(\epsilon) \epsilon d\epsilon = \frac{m}{\pi} \int_0^{\epsilon_F} \epsilon d\epsilon \\ &= \frac{\pi}{4m} \left(\frac{N}{V} \right)^2 \end{aligned} \quad (8)$$

(c) Grand-canonical potential, its relation with energy E , and the pressure p . We start from

$$\begin{aligned} \Omega &= -kT g_s \int g(k) dk \ln(1 + \exp(-\beta(\epsilon_k - \mu))) \\ &= -\frac{2kTV}{2\pi} \int_0^\infty k dk \ln(1 + \exp(-\beta(\epsilon(k) - \mu))) \end{aligned} \quad (9)$$

After integrating by parts, we get that

$$\begin{aligned} \Omega &= -kT g_s \int g(k) dk \ln(1 + \exp(-\beta(\epsilon_k - \mu))) \\ &= -2kT \frac{V}{\pi} \left\{ \left[\frac{k^2}{2} \ln(1 + \exp(-\beta(\epsilon_k - \mu))) \right]_0^\infty + \frac{\beta}{2m} \int dk k^3 \frac{1}{1 + e^{\beta(\epsilon(k) - \mu)}} \right\} \\ &= -\frac{V}{2m\pi} \int dk k^3 \frac{1}{1 + e^{\beta(\epsilon(k) - \mu)}} \end{aligned} \quad (10)$$

The internal energy E_T of a gas of fermions at temperature T is given by

$$\begin{aligned} E_T &= g_s \int dk g(k) \epsilon_k n_F(\epsilon_k, T) \\ &= \frac{V}{2m\pi} \int dk k^3 \frac{1}{1 + e^{\beta(\epsilon(k) - \mu)}} \\ &= -\Omega \end{aligned} \quad (11)$$

where we have compared this with (10) to obtain the equality in the last line.

(d) Adiabatic process.

We can start from the Gibbs - Dunham relation, according to which $\Omega = -pV$. In the previous part of the question, we have proved that in $d = 1$, $\Omega = -E$. Then it follows that $E = pV$. Differentiating this expression we get that

$$\delta E = p\delta V + V\delta p \quad (12)$$

For the adiabatic process, it holds that $\delta E = \delta Q - p\delta V = -p\delta V$, since there is no heat exchanged with the environment. Using this in conjunction with (12) we get that

$$-p\delta V = p\delta V + V\delta p \quad (13)$$

From this it follows that $pV^2 = \text{const}$. From (10) we see that we can rewrite the grand-canonical potential in the form of

$$\Omega = T^2 V f(\mu/T) \quad (14)$$

The entropy per unit volume is given by

$$\begin{aligned} S/V &= -1/V \left(\frac{\partial \Omega}{\partial T} \right)_{V, \mu} \\ &= 2T f(\mu/T) - \mu f'(\mu/T) \end{aligned} \quad (15)$$

From above we conclude that $\frac{S}{L}(\alpha\mu, \alpha T) = \alpha \frac{S}{L}(\mu, T)$, i.e. $\frac{S}{V}$ is a homogeneous function of μ, T of degree 1. Similarly we get that

$$\begin{aligned} N/V &= -1/V \left(\frac{\partial \Omega}{\partial \mu} \right)_{T, \mu} \\ &= -T f'(\mu/T) \end{aligned} \quad (16)$$

From above we conclude that $\frac{N}{V}(\alpha\mu, \alpha T) = \alpha \frac{N}{V}(\mu, T)$, i.e. $\frac{N}{V}$ is a homogeneous function of μ, T of degree 1.

Therefore, it follows that $\frac{S(\mu, T)}{N(\mu, T)} = \frac{S(\alpha\mu, \alpha T)}{N(\alpha\mu, \alpha T)}$ is homogeneous function of degree 0. Since in adiabatic process S is constant, and N is constant, it must follow that $\frac{\mu}{T}$ is constant in adiabatic process. From (16), it follows that $TV = \text{const}$. To find the $pT^{-\delta} = \text{const}$ exponent, we use that $pV^2 = \text{const}$ and $T^2V^2 = \text{const}$. Dividing these leads to $pT^{-2} = \text{const}$. Hence $\delta = 2$.

3. Nematic transition

(10 + 10 + 5 = 25 Punkte)

(a) Nematic order parameter. The free energy is given by:

$$F(\epsilon, \phi) = \frac{1}{2} c_0 \epsilon^2 - \lambda \epsilon \phi + \frac{1}{2} \chi_0^{-1} \phi^2 + \frac{B}{4} \phi^4 \quad (17)$$

Solving for $\frac{\partial F}{\partial \epsilon} = 0$, we get that

$$\epsilon = \frac{\lambda}{c_0} \phi. \quad (18)$$

The equation $\frac{\partial F}{\partial \phi} = 0$ leads to

$$-\lambda \epsilon + \chi_0^{-1} \phi + B \phi^3 = 0 \quad (19)$$

Substituting (18) into (19) leads to, either $\phi = 0$ or

$$\begin{aligned} \phi &= \sqrt{\left(\frac{\lambda^2}{c_0} - \chi_0^{-1} \right) \frac{1}{B}} \\ &= \sqrt{\left(\frac{\lambda^2}{c_0} - A(T - T_0) \right) \frac{1}{B}} \end{aligned} \quad (20)$$

which is a solution iff $\frac{\lambda^2}{c_0} - A(T - T_0) > 0$; otherwise $\phi = 0$

(b) Elastic modulus. First we need to consider the partial derivatives. We have $F = F(\epsilon, \phi)$, and that $\frac{\partial F}{\partial \phi} = \frac{\partial F}{\partial \epsilon} = 0$, which makes ϕ dependent on ϵ . We start from

$$\frac{\delta F}{\delta \epsilon} = \frac{\partial F}{\partial \epsilon} + \frac{\partial F}{\partial \phi} \frac{\partial \phi}{\partial \epsilon} \quad (21)$$

Then using (21) we get that:

$$\delta \left(\frac{\delta F}{\delta \epsilon} \right) = \frac{\partial^2 F}{\partial \epsilon^2} \delta \epsilon + \frac{\partial^2 F}{\partial \epsilon \partial \phi} \delta \phi + \left(\frac{\partial^2 F}{\partial \phi^2} \delta \phi + \frac{\partial^2 F}{\partial \phi \partial \epsilon} \delta \epsilon \right) \frac{\partial \phi}{\partial \epsilon} + \delta \left(\frac{\partial \phi}{\partial \epsilon} \right) \underbrace{\frac{\partial F}{\partial \phi}}_0 \quad (22)$$

Therefore it follows that

$$\frac{\delta^2 F}{\delta \epsilon^2} = \frac{\partial^2 F}{\partial \epsilon^2} + 2 \frac{\partial^2 F}{\partial \epsilon \partial \phi} \frac{\partial \phi}{\partial \epsilon} + \frac{\partial^2 F}{\partial \phi^2} \left(\frac{\partial \phi}{\partial \epsilon} \right)^2 \quad (23)$$

and, using that $\frac{\partial \phi}{\partial \epsilon} = -\frac{\frac{\partial^2 F}{\partial \epsilon \partial \phi}}{\frac{\partial^2 F}{\partial \phi^2}}$ (which was given as a hint in the question) we get that

$$\begin{aligned} c &= \left. \frac{d^2 F}{d\epsilon^2} \right|_{\frac{\partial F}{\partial \epsilon} = \frac{\partial F}{\partial \phi} = 0} \\ &= \frac{\partial^2 F}{\partial \epsilon^2} - \left(\frac{\partial^2 F}{\partial \epsilon \partial \phi} \right)^2 \left(\frac{\partial^2 F}{\partial \phi^2} \right)^{-1} \end{aligned} \quad (24)$$

where we used the hint given in the question going from 2nd into third line. From (17) we find that

$$\begin{aligned} \frac{\partial^2 F}{\partial \epsilon^2} &= c_0 \\ \frac{\partial^2 F}{\partial \epsilon \partial \phi} &= -\lambda \\ \frac{\partial^2 F}{\partial \phi^2} &= \chi_0^{-1} + 3B\phi^2 \end{aligned} \quad (25)$$

Then substituting these in (24) we get that

$$c = c_0 - \frac{\lambda^2}{\chi_0^{-1} + 3B\phi^2} \quad (26)$$

(i) In the normal phase $\phi = 0$ and thus

$$c = c_0 - \frac{\lambda^2}{A(T - T_0)} \quad (27)$$

Shuffling this we can rewrite it in the form

$$c/c_0 = \frac{T - T_c}{T - T_0} \quad (28)$$

where $T_c = T_0 + \frac{\lambda^2}{Ac_0}$

(ii) Nematic phase. In the nematic phase, as we have found in a):

$$\phi = \sqrt{\left(\frac{\lambda^2}{c_0} - A(T - T_0)\right)} \frac{1}{B} \quad (29)$$

Then substituting this into (26) we get

$$c/c_0 = 1 - \frac{\lambda^2}{c_0 \left(\frac{3\lambda^2}{c_0} - 2\chi_0^{-1}\right)} \quad (30)$$

Shuffling this expression we get to

$$\frac{c}{c_0} = \frac{2(T_c - T)}{3T_c - T_0 - 2T} \quad (31)$$

(c) Graph. From the previous part we have seen that

$$\begin{aligned} c/c_0|_{normal} &= \frac{T - T_c}{T - T_0} \\ \frac{c}{c_0}|_{nematic} &= \frac{2(T_c - T)}{3T_c - T_0 - 2T} \end{aligned} \quad (32)$$

From this it follows that at the nematic phase transition $c \rightarrow 0$, and the slopes of c vs T in normal and nematic phases are different, due to different temperature dependencies in (32). From the graph, we read off that $c \rightarrow 0$ at about $T_c = 140K$ (when red/blue curve reach zero).