Skalengesetze

in der Nähe eines Phasenübergangs 2. Ordnung divergiert die Korrelationslänge

$$\xi(T \to T_c) \to \infty$$

$$t = \frac{T - T_c}{T_c} \qquad \xi(t) \sim t^{-\nu}$$

andere physikalische Variablen folgen auch Potenzgesetzen (in der Nähe der Phasenübergangstemp.)

Ordnungsparameter $\quad \phi(t, h=0) \sim |t|^{\beta}$

—— " —— $\quad \phi(0, h) \sim h^{1/\delta}$

Suszeptibilität $\quad \chi(t) \sim \left.\frac{\partial \phi}{\partial h}\right|_{h \to 0} \sim |t|^{-\gamma}$

Wärmekapazität $\quad C(t) \sim t^{-\alpha}$

Suszeptibilität (mittlere) $\quad \chi(|\vec{x}|, t=0) \sim \frac{1}{|\vec{x}|^{d-2+\eta}}$

Sind diese Exponenten voneinander unabhängig??

exponent	Molekularfeld	d=2 Ising	d=3 Ising
α	0	„0"	0.12
β	$\frac{1}{2}$	$\frac{1}{8}$	0.31
γ	1	$\frac{7}{8}$	1.25
ν	$\frac{1}{2}$	1	0.64
δ	3	15	5.0
η	0	1/4	0.04

Molekularfeldtheorie

$$\chi(q,t) = \frac{}{q^2 + t}$$

$$q \to q' = bq$$

b : Skalierungs-faktor

$$\chi(q,t) = b^2 \chi(bq, t b^2)$$

Annahme: die exakte Suszeptibilität erfüllt auch ein entsprechendes Skalengesetz:

$$\chi(q,t) = b^{2-\eta}\, \chi(bq, t b^{y})$$

o. E. d. A. $\quad b: \quad t b^{y} = 1$

$$b = t^{-1/y}$$

$$\Rightarrow \quad \chi(q,t) = t^{-\frac{2-\eta}{y}}\, \chi(q t^{-1/y}, 1)$$

$q=0$

$$\chi(0,t) = t^{-\frac{2-\eta}{y}}\, \chi(0,1)$$

$$\boxed{\gamma = \frac{2-\eta}{y}}$$

$\chi(q,t) \sim f(q\,\xi(t)) \qquad \xi \sim t^{-\nu} \qquad \nu = 1/y$

$$\boxed{\gamma = \nu(2-\eta)}$$ Skalengesetz

entsprechend können wir eine Skalenrelation für die freie Energie postulieren:

$$f(t,h) = b^{-d} f(t b^{1/\nu}, h b^{y_h})$$

Wärmekapazität:

$$c \sim \frac{\partial^2 f}{\partial t^2}$$

$$c(t) = b^{-(d-2/\nu)} \, c(t b^{1/\nu})$$

$$t b^{1/\nu} = 1$$

$$b = t^{-\nu}$$

$$c(t) \sim t^{-(2-\nu d)}$$

$$\boxed{\alpha = 2-\nu d}$$

Ordnungsparameter

$$\phi \sim \frac{\partial f}{\partial h}$$

$$\phi(t,h) = b^{y_h - d}\, \phi(t b^{1/\nu}, h b^{y_h})$$

$$\phi(t,0) \sim t^{\nu(d-y_h)}$$

$$\beta = \nu(d-y_h)$$

$$\phi(0,h) \sim h^{\frac{d}{y_h}-1} \qquad \delta = \frac{y_h}{d-y_h}$$

$$\chi(t) = b^{2y_h-d}\, \chi(t b^{1/\nu})$$

$$2-\eta = 2y_h - d$$

$$\alpha = 2 - d\nu$$

$$\gamma = \nu(2-\eta)$$

$$\beta(1+\delta) = 2-\alpha$$

$$2\beta\delta - \gamma = 2-\alpha$$

Insgesamt gibt es zwei Exponenten, die dann alle weiteren Exponenten bestimmen!!

$$\chi(q,t) = b^{2-\eta}\,\chi(bq, b^{1/\nu} t)$$

die Methode der Renormierungsgruppe

① Gauß'sche Fluktuationen

$$\phi(x) = \phi_0 + \psi(x) \qquad ; \ \phi_0 = \langle \phi \rangle$$

$$Z = \int \mathcal{D}\phi \, e^{-H[\phi]}$$

$$\int (\nabla\phi)^2 d^dx = \int \nabla\phi \cdot \nabla\phi \, d^dx$$

$$H = \frac{1}{2}\int d^dx \, \phi(x)\left(r - \nabla^2\right)\phi(x) + \frac{u}{4}\int d^dx \, \phi^4(x)$$

$$= V\left(\frac{r}{2}\phi_0^2 + \frac{u}{4}\phi_0^4\right) + \frac{1}{2}\int d^dx \, \psi(x)\left(r - \nabla^2 + 3u\phi_0^2\right)\psi(x)$$

$$+ \dots$$

$$\int r \, \underset{\phi_0}{\phi(x)} \, \underset{\psi(x)}{\phi(x)} \, d^dx \Rightarrow r \int \phi_0 \, \psi(x) \, d^dx = r\phi_0 \underbrace{\int \psi(x) d^dx}_{\psi(k\to 0)}$$

$$Z = \int \mathcal{D}\psi \, e^{+V\left(\frac{r}{2}\phi_0^2 + \frac{u}{4}\phi_0^4\right)} e^{-\frac{1}{2}\int d^dx \, \psi(x)\left(r - \nabla^2 + 3u\phi_0^2\right)\psi(x)}$$

$$= e^{V\left(\frac{r}{2}\phi_0^2 + \frac{u}{4}\phi_0^4\right)} \int \mathcal{D}\psi \, \exp\left\{-\frac{1}{2}\int \frac{d^dk}{(2\pi)^d} \, \psi(k)\left(r + 3u\phi_0^2 + k^2\right)\psi(-k)\right\}$$

$$\int \mathcal{D}\psi \; e^{-\frac{1}{2}\sum_k' \psi_k A_k \psi_{-k}}$$

$$= \int \Big(\prod_k d\psi_k\Big)\, e^{-\frac{1}{2}\sum_k \psi_k A_k \psi_{-k}}$$

$$= \prod_k \underbrace{\left(\int_{-\infty}^{\infty} d\psi_k \, e^{-\frac{1}{2}\psi_k A_k \psi_{-k}} \right)}_{\sim \frac{1}{\sqrt{A_k}}}$$

$$\sim \prod_k A_k^{-1/2} = e^{-\frac{1}{2}\sum_k \log A_k}$$

Freie Energie:

$$\frac{F}{V} = \frac{r}{2}\phi_0^2 + \frac{u}{4}\phi_0^4 + \frac{1}{2}\int \frac{d^dk}{(2\pi)^d} \log\left(r + k^2 + 3u\phi_0^2\right)$$

$$\log\left(r+k^2+3u\phi_0^2\right) \simeq \log\left(r+k^2\right) + \frac{3u\phi_0^2}{r+k^2} - \frac{1}{2}\frac{9u^2\phi_0^4}{(r+k^2)^2} + \ldots$$

$$\frac{F}{V} = \frac{F_0}{V} + \frac{r'}{2}\phi_0^2 + \frac{u'}{4}\phi_0^4$$

$$\boxed{\begin{aligned} r' &= r + 3u \int \frac{d^d k}{(2\pi)^3} \frac{1}{r+k^2} \\ u' &= u - 9u^2 \int \frac{d^d k}{(2\pi)^3} \frac{1}{(r+k^2)^2} \end{aligned}}$$

In der Nähe des Phasenübergangs $\quad r \to 0$

$$\int \frac{d^d k}{k^2} \sim \int_0^{\Lambda} \frac{k^{d-1}\,dk}{k^2} \qquad \Lambda \sim \frac{2\pi}{a_0}$$

(cut-off)

$$\sim \begin{cases} \Lambda^{d-2} & d > 2 \\ \infty & d \le 2 \end{cases}$$

ultraviolette Verhalten

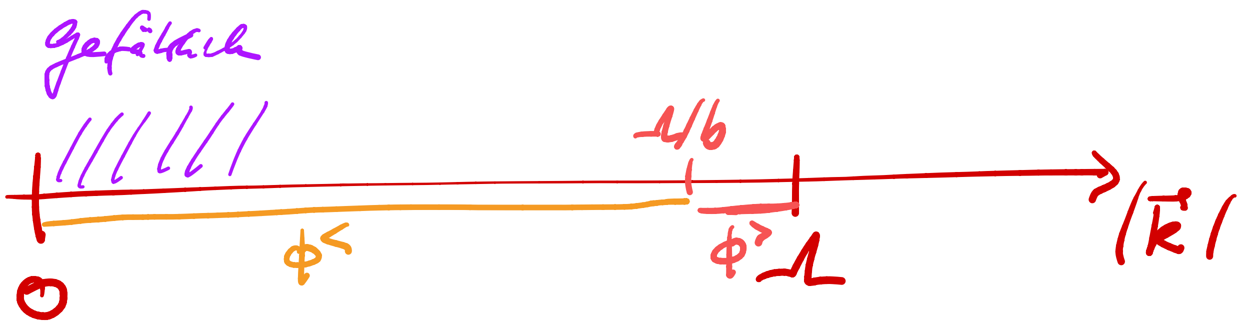
infrarote Divergenz

$$\int \frac{d^d k}{k^4} \sim \int_0^{\Lambda} \frac{k^{d-1}\,dk}{k^4} \sim \begin{cases} \Lambda^{d-4} & d > 4 \\ \infty & d \le 4 \end{cases}$$

2. langsame + schnelle Variablen

$$\phi(\vec{x}) \to \phi(\vec{k}) = \int d^d x \, e^{i\vec{k}\cdot\vec{x}} \phi(\vec{x})$$

Problem sind infrarot-Divergenzen



$$Z = \int \mathcal{D}\phi \, e^{-H[\phi]}$$

$$\phi_k^< : \quad \phi_k \quad 0 < k < \Lambda/b$$

$$\phi^> : \quad \phi_k \quad \Lambda/b < k < \Lambda$$

$$Z = \int \mathcal{D}\phi^< \mathcal{D}\phi^> \, e^{-H[\phi^<, \phi^>]}$$

$$= \int \mathcal{D}\phi^< \, e^{-H'[\phi^<]}$$

$$H[r,u] \longrightarrow H'[r,u] = H[r',u'] + \text{const.}$$

Analyse für $u=0$

$$H = \frac{1}{2}\int^{\Lambda} \frac{d^dk}{(2\pi)^d}\,(r+k^2)\,\phi_k\,\phi_{-k}$$

$$H' = \frac{1}{2}\int^{\Lambda/b} \frac{d^dk}{(2\pi)^d}\,(r+k^2)\,\phi_k\,\phi_{-k}$$

$$k' = bk \qquad \phi'(k') = b^{-\zeta}\phi(k)$$

$$= \frac{1}{2}\,b^{-2-d+2\zeta}\int^{\Lambda} \frac{d^dk'}{(2\pi)^d}\,(rb^2+k'^2)\,\phi'(k')\,\phi'(-k')$$

$$\zeta = \frac{d+2}{2}$$

$$H' = \frac{1}{2}\int^{\Lambda} \frac{d^dk}{(2\pi)^d}\,(rb^2+k^2)\,\phi_k\,\phi_{-k}$$

$$r \longrightarrow r' = rb^2$$

mit Wechselwirkungseffekten:

$$\boxed{r' = r b^2 + 3u \int_{\Lambda/b}^{\Lambda} \frac{d^d k}{(2\pi)^d} \frac{1}{r + k^2}}$$

Wechselwirkungsterm:

$$H_u = \frac{u}{4} \int d^d x \, \phi(x)^4$$

$$= \frac{u}{4} \int^{\Lambda} \frac{d^d k_1 \, d^d k_2 \, d^d k_3}{(2\pi)^3} \phi_{k_1} \phi_{k_2} \phi_{k_3} \phi_{-(k_1+k_2+k_3)}$$

$$\downarrow$$

$$H_u = \frac{u}{4} \int^{\Lambda/b} \frac{d^d k_1 \, d^d k_2 \, d^d k_3}{(2\pi)^3} \phi_{k_1} \phi_{k_2} \phi_{k_3} \phi_{-(k_1+k_2+k_3)}$$

$$k' = bk \qquad \phi' = b^{-\zeta} \phi$$

$$= \frac{u}{4} b^{-d \cdot 3 + 4\zeta} \int^{\Lambda} \frac{dk_1 \dots dk_3}{(2\pi)^3} \phi_{k_1} \dots \phi_{-(k_1+k_2+k_3)}$$

$$4\zeta - 3d = 2(d+2) - 3d = 4 - d$$

$$\boxed{u' = u\, b^{4-d} - g u^2 \int_{\Lambda/b}^{\Lambda} \frac{d^d k}{(2\pi)^d} \frac{1}{(r+k^2)^2}}$$